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The extraction of hadronic parameters from experiments on pionium

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Abstract

We show how experimental values of the lifetime of the $1s$ level of pionium and of the difference between the energies of the $2s$ and $2p$ levels yield values of the elements a_{oc} and a_{cc} respectively of the s-wave scattering matrix for the two-channel $(\pi^+\pi^-, \pi^0\pi^0)$ system at the $\pi^+\pi^-$ threshold. We then develop a method, using energy independent hadronic potentials which reproduce the best available pion-pion phase shifts up to 500 MeV total energy in the c.m. frame, for obtaining the values of the isospin invariant quantities $a_0^2 - a_0^0$ and $2a_0^0 + a_0^2$ from a_{oc} and a_{cc} respectively. We emphasise that the isospin invariant scattering lengths a_0^0 and a_0^2 universally used in the literature cannot be considered to be purely hadronic quantities.

I. INTRODUCTION

The $\pi^+\pi^-$ atom (pionium) has become of considerable experimental and theoretical interest. Such atoms have been clearly observed at Dubna [1] and a lower limit obtained for the lifetime τ_{1s} of the lowest $(1s)$ level [2]. A large collaboration is engaged in an experiment at CERN to measure τ_{1s} with an accuracy of about 10% [3]. In the proposal for this experiment the possibility of measuring the difference $\Delta W = W_{2s} - W_{2p}$ between the energies of the $2s$ and $2p$ levels is also considered. The method was proposed by Nemenov [4] and its feasibility can be tested using the same experimental setup as for the measurement of τ_{1s}.

A complementary approach has been suggested in Ref. [5] by a group at IUCF. They have considered the possibility of measuring the ratio of the rates for the decay of $\pi^+\pi^-$, either in ns bound states or low-lying continuum states, into $\pi^0\pi^0$ and $\gamma\gamma$. The known rate for the almost purely electromagnetic $\gamma\gamma$ decay would then yield the rate for the $\pi^0\pi^0$ decay. Ref. [5] indicates however that a first search for a suitable reaction with $\pi^+\pi^-$ pairs in the final state has not yielded promising results. We therefore concentrate in this paper on the results expected to come from the CERN experiment.

A measurement of τ_{1s} immediately yields the width of the $1s$ level via the relation

$$\Gamma_{1s}^{tot}(\text{eV}) = 0.658\ 212\ 2[\tau_{1s}(\text{fs})]^{-1}. \tag{1}$$

The partial width for decay to $\gamma\gamma$ is very small, and can be derived from QED for bosons [6]. The result is

$$\Gamma_{1s}^{\gamma\gamma} = \frac{1}{4}\alpha^5\mu_c = 7.2 \times 10^{-4}\text{eV}, \tag{2}$$

α being the fine structure constant and μ_c the mass of a charged pion. Subtracting Eq. (2) from Eq. (1) gives the value of $\Gamma_{1s}^{\pi^0\pi^0}$, which we shall henceforth abbreviate to Γ. Since Γ is expected to be about 0.177 eV and will be measured with an accuracy of around 10%, no refined corrections to the result in Eq. (2) are needed.

We expect then that values of Γ and, later, ΔW will become available from experiments on pionium. In Section 2 we shall go briefly through the formalism for pionium, which is identical with that given by Rasche and Woolcock [7] for pionic hydrogen. We shall give the explicit formulae for a_{oc}(fm) and a_{cc}(fm) in terms of Γ(eV) and ΔW(eV) respectively. The matrix

$$\mathbf{a} = \begin{pmatrix} a_{cc} & a_{oc} \\ a_{oc} & a_{oo} \end{pmatrix} \tag{3}$$

is the s-wave scattering matrix for the two-channel $(\pi^+\pi^-, \pi^0\pi^0)$ system at the $\pi^+\pi^-$ threshold. The subscript c refers to the $\pi^+\pi^-$ channel, o to the $\pi^0\pi^0$ channel.

The challenge we are addressing is to connect these numerical results, derivable in principle from experiments on pionium, with our knowledge of pion-pion scattering at low energies (by which we shall mean that the total energy W in the c.m. frame is less than about 500 MeV). This knowledge comes from two sources, experiments in which pion-pion pairs in the final states of various reactions are studied, and chiral perturbation theory (χPT), which gives a low-energy representation of the $\pi\pi$ scattering amplitudes based on QCD. The difficulty of the problem lies in the subtle interplay of the two interactions present, the hadronic and the electromagnetic. The values of a_{oc} and a_{cc} which come from pionium experiments are not due to the hadronic interaction alone: they are affected by the electromagnetic interaction. On the other hand, χPT as it has been developed so far, with the quark mass difference neglected and only the QCD lagrangian used, is a hadronic interaction theory which is charge independent (isospin invariant). Moreover, experiments above the $\pi^+\pi^-$ threshold are analysed to give what are quoted as phase shifts for states of definite isospin (δ_0^0 and δ_0^2 for ℓ=0, I=0 and 2; δ_1^1 for ℓ=1, I =1;$\cdots$). In spite of this, as we shall see, neither χPT as it is presently applied, nor the analysis of experiments, completely disentangles the electromagnetic interaction from the hadronic. Nevertheless it is clear that electromagnetic corrections need to be made, to relate the values of a_{oc} and a_{cc} obtained from pionium to the values of the isospin invariant s-wave scattering lengths a_0^0 and a_0^2 obtained from above threshold experiments and χPT.

One approach has been made in a recent preprint by Maltman and Wolf [8], who attempt to include the electromagnetic interaction as an additional symmetry breaking term in χPT in order to calculate the corrections to the purely hadronic $\pi\pi$ scattering lengths. At this stage of their work too few details are given to fully evaluate its consequences for pionium. In this paper we use a potential model, considered previously by Moor, Rasche and Woolcock [9], to calculate the electromagnetic corrections described above.

Moor et al. [9] attempted to calculate $a_{oc} - \frac{\sqrt{2}}{3}(a_0^2 - a_0^0)$ by distinguishing between the physical situation and the purely hadronic situation in which the electromagnetic interaction is switched off ($\alpha = 0$). In the first, the charged and neutral pions have their physical masses μ_c and μ_o respectively and the hadronic and electromagnetic interactions are present. In the second, there is only the hadronic interaction and, in accordance with current opinion [10–12], it is assumed that all the pions have almost the same mass, which is extremely close to μ_o , and that there is practically no isospin breaking of the hadronic scattering amplitudes. Ref. [9] connects these two situations by means of a potential model. For the purely hadronic situation, all the pions were taken to have the mass μ_o and only a charge independent hadronic potential was present. Simple square well potentials were used in a relativistic Schrödinger equation in order to reproduce at the energy $W = 2\mu_o$ the purely hadronic scattering lengths $a_0^I, I = 0, 2$, obtained from χPT. For the physical situation a two-channel s-wave relativistic Schrödinger equation was used, with physical masses and (relativistic) c.m. momenta at a fixed energy, the potential matrix being the sum of a hadronic potential and a Coulomb potential for extended charge distributions. In the spirit of χPT it was assumed that the mass difference is the only source of symmetry breaking, so that the hadronic potential for the physical situation was taken to be exactly the same as the charge independent potential for the purely hadronic situation. The two-channel equation for the physical situation was then solved to obtain a_{oc} at $W = 2\mu_c$, and thus the difference $a_{oc} - \frac{\sqrt{2}}{3}(a_0^2 - a_0^0)$.

There are two serious problems with the work in Ref. [9]. First, the purely hadronic scattering lengths a_0^0 and a_0^2 were obtained by inserting the mass μ_o into the χPT equations

of Gasser and Leutwyler [10]. However some constants in those equations are fixed by using experimental data extrapolated to the $\pi^+\pi^-$ threshold $W = 2\mu_c$. This is an inconsistent procedure. In the present state of our knowledge it is not possible to obtain purely hadronic scattering lengths at the purely hadronic threshold $W = 2\mu_o$. One is prevented by the fact that, in the analysis of experiments and in the practical use of χPT, the $\pi\pi$ system at the hadronic level is treated as an isospin invariant system with all the pions having the mass μ_c. On the basis of this assumption charge independent phase shifts are obtained from experiments. This information is then used, in conjunction with the χPT equations (the pion mass being taken as μ_c), to generate s-wave phase shifts δ_0^0 and δ_0^2 for $W \geq 2\mu_c$. Sets of these phases have been provided to us by Gasser [13]. We note here that their behaviour at low energies is different from that in the fits of Froggatt and Petersen [14] and Lohse et al. [15]. Using the expansion

$$\tan \delta_0^I / q_c = a_0^I + \alpha_0^I q_c^2 + \cdots, I = 0, 2, \tag{4}$$

the χPT equations show clearly that $\alpha_0^0 > 0, \alpha_0^2 < 0$, so that the curvatures α_0^I have the same signs as the respective scattering lengths a_0^I. For this reason, we have reservations about the potentials constructed by Sander, Kuhrts and von Geramb [16], based on the fits of Refs. [14,15].

The second problem in the work of Ref. [9] is that simple square well potentials were used. To obtain the values of both a_0^I and α_0^I given by χPT, it is necessary for such potentials to be energy dependent. Further, Moor et al. calculated the difference between the purely hadronic scattering matrix at $W = 2\mu_o$ and the physical scattering matrix at $W = 2\mu_c$ without taking into account the variation of their potentials with energy. This particular problem no longer arises in the present work, where we shall calculate electromagnetic corrections at $W = 2\mu_c$. Nevertheless, there remain strong reasons for desiring an energy independent hadronic potential matrix

$$\mathbf{V^{had}} = \begin{pmatrix} \frac{2}{3}V_0^0 + \frac{1}{3}V_0^2 & \frac{\sqrt{2}}{3}(V_0^2 - V_0^0) \\ \frac{\sqrt{2}}{3}(V_0^2 - V_0^0) & \frac{1}{3}V_0^0 + \frac{2}{3}V_0^2 \end{pmatrix}. \tag{5}$$

If such potentials V_0^I can be found, the representation of the strong interaction by a potential matrix which remains unchanged when the electromagnetic interaction is switched on is placed on much firmer ground. The assumption that the $\pi\pi$ system at the hadronic level can be treated as an isospin invariant system with all the pions having the mass μ_c then translates into a two-channel Schrödinger equation for which the equations decouple in the isospin basis. The mass μ_c plays the role of a convenient reference mass and we emphasise that the scattering lengths a_0^I obtained from the potentials V_0^I by using a Schrödinger equation with this mass are isospin invariant but not purely hadronic quantities.

There is a further question about how to construct energy independent potentials V_0^I. If the phase shifts δ_0^I were perfectly known and the process remained elastic at all energies, it would be possible by inversion to construct potentials V_0^I. However, there are substantial uncertainties in the values of δ_0^I (and electromagnetic corrections in the sense of Ref. [17] have not been made), there are inelastic processes at sufficiently high energy and, most important of all, there is no reason to believe that the hadronic interaction can be represented by potentials except at low energies. We therefore do not use the results of Ref. [16] but instead find potentials of simple shape, characterised by a small number of parameters, which can be fixed so that the potentials reproduce the phase shifts well up to $W = 500$ MeV. In fact double square wells with in all four parameters work quite satisfactorily. A sum of two gaussians or of two lorentzian shapes were also tried, but the fit to the phase shifts could not be further improved and it was more difficult to ensure that the potentials had no bound states.

The double square well energy independent potentials V_0^I are fed into the one-channel Schrödinger equations

$$\left(\frac{d^2}{dr^2} + q_c^2 - \mu_c V_0^I(r)\right) u(r) = 0, I = 0, 2, \tag{6}$$

and the ranges and depths of the two wells adjusted to obtain the best possible fit to the phase shifts δ_0^I up to $W = 500$ MeV. Eq. (6) also yields the scattering lengths a_0^I and we denote by $\tilde{\mathbf{a}}$ the matrix

$$\tilde{\mathbf{a}} = \begin{pmatrix} \frac{2}{3}a_0^0 + \frac{1}{3}a_0^2 & \frac{\sqrt{2}}{3}(a_0^2 - a_0^0) \\ \frac{\sqrt{2}}{3}(a_0^2 - a_0^0) & \frac{1}{3}a_0^0 + \frac{2}{3}a_0^2 \end{pmatrix}. \tag{7}$$

For the physical situation we have the two-channel Schrödinger equation with physical masses

$$\left(\mathbf{1}_2\frac{d^2}{dr^2} + \mathbf{Q}^2 - 2\mathbf{M}\mathbf{V}^{had}(r) - 2\mathbf{M}\mathbf{V}^{C}(r)\right)\mathbf{u}(r) = 0, \tag{8}$$

where $\mathbf{V^{had}}$ is given by Eq. (5) and

$$\mathbf{Q} = \begin{pmatrix} q_c & 0 \\ 0 & q_o \end{pmatrix}, 2\mathbf{M} = \begin{pmatrix} \mu_c & 0 \\ 0 & \mu_o \end{pmatrix}, \mathbf{V}^C(r) = \begin{pmatrix} V^C(r) & 0 \\ 0 & 0 \end{pmatrix}.$$

The relativistic c.m. momenta q_c, q_o in the two channels are calculated from the same value of W; $V^C(r)$ is the Coulomb potential between extended charge distributions. The coupled equations (8) are integrated numerically to give the scattering matrix $\mathbf{K}$ as a function of W. The details are exactly the same as those given in Section 3 of Ref. [18] for the coupled $(\pi^- p, \pi^0 n)$ system; the matrix $\mathbf{A}$ introduced in Eq. (65) of that paper we now denote by $\mathbf{K}$, in accordance with the altered notation of Ref. [7]. Extrapolation to $W = 2\mu_c$ gives the threshold scattering matrix $\mathbf{a} = \mathbf{K}(2\mu_c)$ of Eq. (3) and thus the difference $\tilde{\mathbf{a}} - \mathbf{a}$, which is the final goal of our calculation. The values of a_{oc} and a_{cc} obtained from experiments on pionium can then be corrected to give values of $(a_0^2 - a_0^0)$ and $(2a_0^0 + a_0^2)$ which can be compared with the values obtained from above threshold experiments and χ PT. In Section 3 we shall give further details of the calculations sketched above, and the numerical results for the electromagnetic corrections.

II. DETERMINATION OF SCATTERING PARAMETERS FROM PIONIUM DATA

The results in this section for the positions and widths of the pionium states of interest come from the formalism of Rasche and Woolcock [7], which uses analytic continuation of

the scattering matrix below the threshold for a channel containing two oppositely charged hadrons. These formal results have been confirmed by determining the position and width directly, by studying the behaviour of the phase shift for the (open) $\pi^0\pi^0$ channel. We also remark that the treatment of Ref. [7] is complete, and includes what are often (misleadingly) called hadronic effects on the wavefunction at the origin. The results of Ref. [16] for the ns states of pionium, which are quite different from ours, are known to be wrong. New results for the $1s$ state are given in a recent preprint by Sander and von Geramb [19], but no comment is made on the dramatic difference between these new results and those of Ref. [16].

The width Γ for the decay of pionium in the $1s$ state to $\pi^0\pi^0$ is given in Eqs. (2.10) and (2.13) of Ref. [9] as

$$\Gamma = \frac{8}{B}|1 + p_1(1s)\varepsilon_0(1s) + \cdots|^2[-W_C(1)]q_o a_{oc}^2/(1 + q_o^2 a_{oo}^2), \tag{9}$$

where

$$B = \left(\frac{1}{2}\alpha\mu_c\right)^{-1}, -W_C(1) = \frac{1}{4}\alpha^2\mu_c,$$

$$p_1(1s) = \frac{1}{2}(1+\gamma), \mathrm{Re}\varepsilon_0(1s) = 4a_{cc}/B,$$

α being the fine structure constant and γ Euler's constant. The treatment of the $\pi^- p$ atom in Ref. [7], from which the result, Eq. (9), is taken, takes account neither of the relativistic correction to the bound state position nor of the presence of the vacuum polarisation potential in the calculation of Γ. The first is of no consequence at the level of accuracy with which we are concerned. A full treatment of the second has been worked out by Oades, Rasche and Woolcock [20] and leads to a small modification of Eq. (9). The width in the presence of vacuum polarisation is increased by the factor $(1 + 3.08 \times 10^{-3})$; this result has been obtained both analytically and by a computer calculation of the width with and without the vacuum polarisation potential.

Strictly speaking, the elements of the scattering matrix appearing in Eq. (9) should be the elements of the K-matrix at the bound state position, not at the $\pi^+\pi^-$ threshold. The

only significant effect is on K_{oc}^2; the small modification to Eq. (9) is estimated in Ref. [9] and amounts to a factor $(1 - 2.4 \times 10^{-5})$.

Eq. (9) requires the values of a_{cc} and a_{oo}, for which we use the final numbers from Section 3,

$$a_{cc} = 0.148 \text{ fm}, \ a_{oo} = 0.053 \text{ fm}.$$

With the other numbers

$$B = 387.489 \text{ fm}, \quad -W_C(1) = 1858.073 \text{ eV},$$

$$q_o = 35.509 \text{ MeV}, \quad p_1(1s) = 0.788608,$$

$$\hbar c = 197.327053 \text{ MeV fm},$$

we obtain the result

$$a_{oc}(\text{fm}) = 0.3802[\Gamma(\text{eV})]^{\frac{1}{2}}. \tag{10}$$

The uncertainty in the conversion constant of Eq. (10) may be roughly estimated as about 0.02%. Most of it comes from the uncertainty in the value of a_{cc} used to calculate $\text{Re}\varepsilon_0(1s)$. Since the expected experimental accuracy in $\Gamma^{\frac{1}{2}}$ is about 5%, the uncertainty in the conversion constant is of academic interest only.

The difference ΔW between the energies of the $2s$ and $2p$ levels has contributions from the relativistic corrections, the vacuum polarisation potential and the presence of the hadronic interaction. From the results of Austen and de Swart [21] the relativistic corrections to the positions of the $2s$ and $2p$ states are

$$-37\mu_c\alpha^4/1024 = -0.014 \text{ eV}(2s),$$

$$-31\mu_c\alpha^4/3072 = -0.004 \text{ eV}(2p).$$

Using a vacuum polarisation potential modified at short distances to take account of the extended charge distributions of the pions, the corresponding shifts in the positions of the $2s$ and $2p$ states are

$$-0.110 \text{ eV}(2s), -0.004 \text{ eV}(2p).$$

Thus

$$\Delta W^{had}(\text{eV}) = \Delta W(\text{eV}) + 0.116, \tag{11}$$

where ΔW^{had} is the difference due to the hadronic interaction.

For the shifts in the positions of the $2s$ and $2p$ states due to the hadronic interaction we again appeal to the results of Ref. [7]. For the $2p$ state the shift is

$$\left(-\frac{1}{16}\alpha^2\mu_c\right)\left(\frac{3a_{cc}(\ell=1)}{2B^3}\right) \approx 10^{-6}\text{eV},$$

which is completely negligible. Thus ΔW^{had} comes entirely from the shift in the $2s$ state,

$$\Delta W^{had} = \left(-\frac{1}{16}\alpha^2\mu_c\right)\left(\frac{2a_{cc}}{B}\right)\left(1 + p_1(2s)\frac{2a_{cc}}{B} + \cdots\right), \tag{12}$$

where

$$p_1(2s) = \ln 2 + \gamma - \frac{1}{2} = 0.770363.$$

It follows from Eq. (12) that

$$a_{cc}(\text{fm}) = -0.4168\Delta W^{had}(\text{eV}). \tag{13}$$

From our present knowledge of a_{cc} we expect that ΔW^{had} will be close to -0.355 eV. The experimenters therefore face the difficult task of measuring with reasonable accuracy a difference ΔW which is about -0.47 eV, compared with the position of the n=2 pure Coulomb levels 464.52 eV below the $\pi^+\pi^-$ threshold.

III. ELECTROMAGNETIC CORRECTIONS TO THE THRESHOLD SCATTERING MATRIX

The final step is to calculate the difference $\tilde{\mathbf{a}} - \mathbf{a}$, where $\tilde{\mathbf{a}}$ and $\mathbf{a}$ are defined in Eqs. (3) and (7) respectively. As explained in Section 1, the first step is to construct energy independent double square well potentials, using the Schrödinger equation 5, which reproduce as accurately as possible up to 500 MeV the phase shifts δ_0^0 and δ_0^2 provided by Gasser [13]. There are inner and outer ranges of the potentials to be varied, and it was found that the best fits are obtained with ranges

$$r_1 = 0.3 \text{ fm}, \ r_2 = 1.5 \text{ fm}.$$

The full potentials which we used to obtain the final results are, to 4 significant figures,

$$\begin{aligned} V_0^0(r) &= -6824 \text{ MeV}, \ \ 0 \leq r < 0.3 \text{ fm}, \\ &= +102.0 \text{ MeV}, \ \ 0.3 \text{ fm} \leq r < 1.5 \text{ fm}, \\ &= 0, \ \ r \geq 1.5 \text{ fm}; \end{aligned} \tag{14}$$

$$\begin{aligned} V_0^2(r) &= +39980 \text{ MeV}, \ \ 0 \leq r < 0.3 \text{ fm}, \\ &= -53.57 \text{ MeV}, \ \ 0.3 \text{ fm} \leq r < 1.5 \text{ fm}, \\ &= 0, \ \ r \geq 1.5 \text{ fm}. \end{aligned} \tag{15}$$

The corresponding scattering lengths, which agree with those obtained by extrapolating the input phase shifts to $W = 2\mu_c$, are

$$a_0^0 = 0.2883 \text{ fm}\,, \ a_0^2 = -0.0617 \text{ fm}\,, \tag{16}$$

and the matrix $\tilde{\mathbf{a}}$ of Eq. (7) is

$$\tilde{\mathbf{a}} = \begin{pmatrix} 0.1716 \text{ fm} & -0.1650 \text{ fm} \\ -0.1650 \text{ fm} & 0.0550 \text{ fm} \end{pmatrix}. \tag{17}$$

Fig. 1 shows that the double square well potentials of Eqs. (14) and (15) reproduce the phase shifts of Ref. [13] remarkably well. Not surprisingly, this agreement deteriorates above 500 MeV but this is of no concern. The simple effective potentials clearly represent the hadronic interaction very well over an energy range which is quite sufficient for our purpose.

The scattering lengths (Eq. (16)) come predominantly from the inner parts of the potentials in Eqs. (14) and (15) and the inner radius of 0.3 fm is approximately the smallest radius used by Moor et al. [9] for their single square well potentials. The use of the much larger outer radius of 1.5 fm with the small outer potential, is a simple way of getting good fits to the phase shifts. The scattering lengths in Eq. (16) differ from those of Ref. [9] mainly because the reference mass used in the χPT equations is μ_c, as explained in Section 1.

The Coulomb potential $V^C(r)$ for extended charge distributions was taken as the potential between uniformly charged spheres, exactly as in Ref. [9] (last paragraph of Section 4). We have checked that using gaussian charge distributions instead makes no difference to the scattering matrix in Eq. (18) at the level of accuracy given.

With the input potentials $V^C(r)$ just described and $\mathbf{V^{had}}$ obtained from Eqs. (5), (14) and (15), the threshold scattering matrix arising from Eq. (8) is

$$\mathbf{a} = \begin{pmatrix} 0.1479 \text{ fm} & -0.1599 \text{ fm} \\ -0.1599 \text{ fm} & 0.0533 \text{ fm} \end{pmatrix}. \tag{18}$$

The desired electromagnetic corrections are therefore

$$\tilde{a}_{cc} - a_{cc} = 0.0237 \text{ fm}\,, \ \tilde{a}_{oc} - a_{oc} = -0.0051 \text{ fm}, \tag{19}$$

or, with the definitions

$$\tilde{a}_{cc} = (1 + \Delta_{cc})a_{cc}, \ \tilde{a}_{oc} = (1 + \Delta_{oc})a_{oc},$$

$$\Delta_{cc} = 0.160, \ \Delta_{oc} = 0.032. \tag{20}$$

It is of interest to note that the contribution to the corrections from the pion mass difference is far greater than that from the Coulomb potential. The detailed numerical results are given in Table 1.

On the basis of many calculations which we have made using different scattering lengths a_0^0 and a_0^2 as the starting values, we have found that the relative correction Δ_{oc} is much less sensitive to these starting values than is the difference $\tilde{a}_{oc} - a_{oc}$. However, the difference $\tilde{a}_{cc} - a_{cc}$ is less sensitive than Δ_{cc}. The uncertainties in the values of $\tilde{a}_{cc} - a_{cc}$ and Δ_{oc} need to be estimated. Even though some fine details of the potentials have no doubt been missed, we are confident that the use of double square well potentials introduces a negligible error. We have estimated the uncertainties due to the variation of the hadronic potentials within limits which correspond to the uncertainties in the input phase shifts and in our fitting procedure. This effectively includes the uncertainties in the input scattering lengths. The final results are

$$\tilde{a}_{cc} - a_{cc} = 0.024 \pm 0.002 \text{ fm}, \ \Delta_{oc} = 0.032 \pm 0.002, \tag{21}$$

where we have kept the number of significant digits appropriate to the size of the estimated errors. The development of two-loop χPT [22] will result in slightly different phase shifts, but the resulting changes in $\tilde{a}_{cc} - a_{cc}$ and in Δ_{oc} will lie within the uncertainities given in Eq. (21).

The values of a_{oc} and a_{cc} obtained from experimental data on pionium must be corrected using Eq. (21) , and the resulting values of $\tilde{a}_{oc}$ and $\tilde{a}_{cc}$ compared with the values of a_0^0 and a_0^2 obtained from a combination of $\pi\pi$ scattering data at energies above $W = 2\mu_c$ and the constraints of χPT. If this comparison should eventually reveal a significant discrepancy, it will then be necessary to look again at what is known about $\pi\pi$ scattering above the $\pi^+\pi^-$ threshold.

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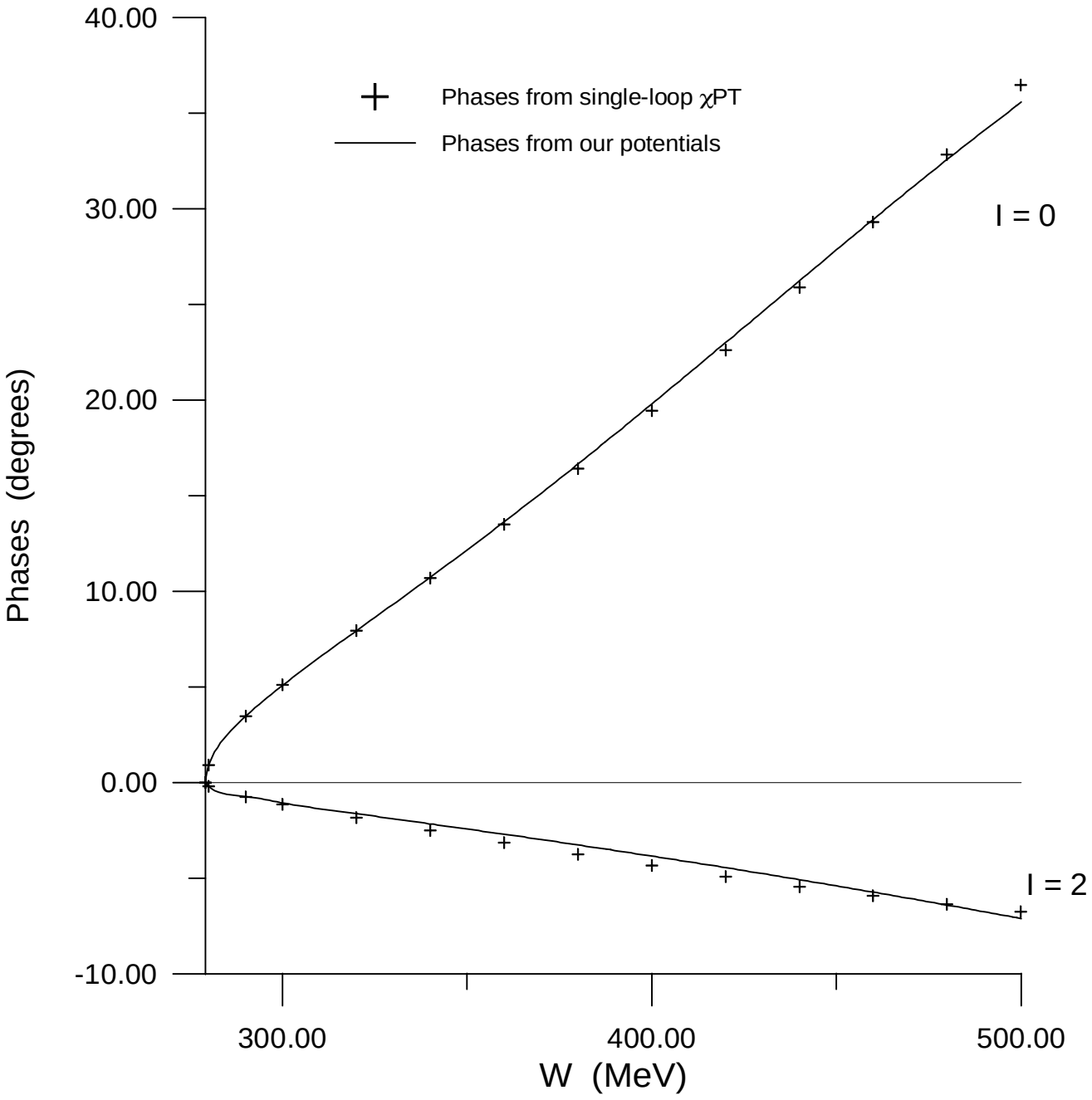
FIGURES

FIG. 1. The $I = 0$ and $I = 2$ $\pi\pi$ s-wave phase shifts δ_0^0 and δ_0^2 given by χPT (Ref. [13]) and by the double square wells of Eqs. (14) and (15).

TABLES

TABLE I. The separate effects of the pion mass difference and of the Coulomb potential on the elements of the scattering matrix at the $\pi^+\pi^-$ threshold (all results in fm)

	a_{cc}	a_{oc}	a_{oo}
hadronic	0.1716	−0.1650	0.0550
with Coulomb potential only	0.1719	−0.1662	0.0561
with mass difference only	0.1478	−0.1589	0.0522
fully corrected	0.1479	−0.1599	0.0533


40.00
30.00
20.00
10.00
0.00
-10.00
Phases (degrees)
Phases from single-loop χPT
Phases from our potentials
I = 0
I = 2
300.00
400.00
500.00
W (MeV)